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A class of double-well like potentials in 1 + 1 dimensions via SUSY QM

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Abstract. The connection between the soliton solutions for potentials in 1 + 1 dimensions and supersymmetry in the context of non-relativistic quantum mechanics is explored. Examining the equivalence between a supersymmetric partner and the stability equation of the ϕ^6 model, which is a Schrödinger-like equation for the normal modes, we construct two new supersymmetric partners, which leads us to a new class of isospectral potentials in quantum mechanics and a new class of potential nonlinear models in bidimensional spacetime. We show that such new potentials in 1 + 1 dimensions are double-well like potentials.

1. Introduction

In a relativistic system of a real scalar field a soliton is a static, non-singular, *classically stable* finite localized energy solution of the equation of motion, which has been examined for the $\lambda\phi^4$ [1–4] and ϕ^6 [5] models and investigations of two [6] and three [7] coupled scalar fields have been made in the context of the stability equations. We implement the well known Bogomol'nyi condition [8] to find the static field configurations and to make the connection with an isospectral Hamiltonian for the ϕ^6 model.

The SUSY algebra has also been applied to construct a variety of new one-parameter families of isospectral supersymmetric partner potentials [9, 10] in quantum mechanics (QM) which are phase-equivalent [11]. Recently, the connection between SUSY QM and the topological and non-topological solitons has been established [12–14].

The formalism of SUSY QM and its applications, including its connection with isospectral potentials, have recently been studied in the literature [15] (for a review see [16]). Recently, a technique to generate exactly-solvable Schrödinger equations by using second-order shift operators was presented and, as an example, a two-parameter family of exactly-solvable Hamiltonians, which contains the Abraham–Moses potential as a particular case has been considered [17].

In this work we construct a new class of isospectral potentials in 1 + 1 dimensions field theory by an application of SUSY QM for the soliton of a self-interacting a real scalar field in 1 + 1 dimensions in the ϕ^6 model.

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This work is organized as follows. In section 2, from the Hamiltonian-like operator associated with the stability equation of the ϕ^6 model we implement the SUSY QM algebra. In section 3 we consider a second SUSY transformation based on a general unnormalizable solution to find a new class of potentials in non-relativistic QM. In section 4 we construct new potentials in field theory which are a new class of double-well like potentials in 1 + 1 dimensions. Section 5 contains the conclusion.

2. SUSY QM and the ϕ^6 model

Let us begin with some known results on the soliton solution in the ϕ^6 model. Using the natural system of units ($c = \hbar = 1$), the Lagrangian density for the ϕ^6 model in 1 + 1 dimensions is written as

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 - V(\phi) \tag{1}$$

where

$$V(\phi) = \frac{1}{2}\lambda^2\phi^2\left(\phi^2 - \frac{\mu}{\lambda}\right)^2 \qquad \mu, \lambda > 0. \tag{2}$$

Here, the well-behaved potential $V(\phi)$ is a positive semidefinite function of ϕ, which possesses a discrete symmetry ($V(\phi) = V(-\phi)$) and three minima to enable a soliton solution to exist.

The equation of motion for a soliton $\phi = \phi_s(x)$ becomes

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi_s(x) + \lambda^2\phi_s\left(\phi_s^2 - \frac{\mu}{\lambda}\right)\left\{3\phi_s^2 - \frac{\mu}{\lambda}\right\} = 0. \tag{3}$$

Since the potential $V(\phi)$ is positive it satisfies the well known Bogomol'nyi condition [8]

$$\frac{\mathrm{d}}{\mathrm{d}x}\phi_s = \pm\sqrt{2V(\phi_s)} = \pm\lambda\phi_s\left(\phi_s^2 - \frac{\mu}{\lambda}\right) \tag{4}$$

where the solution with the minus sign represents the soliton $\phi_s(x)$ and the one with the plus sign the antisoliton. The soliton solution is given by

$$\phi_s(x) = \sqrt{\frac{1}{2}\frac{\mu}{\lambda}[1 + \tanh[\mu(x + x_0)]]}. \tag{5}$$

Note that if $\mu > 1$ then the sign of λ identifies different solutions. Since $\tanh y \to \pm 1$ as $y \to \pm\infty$ it is easy to verify the following boundary conditions: $\phi_s(x) \to \phi_{\mathrm{vacuum2}}(x) = 0$, as $x \to -\infty$ and $\phi_s(x) \to \phi_{\mathrm{vacuum3}}(x) = \sqrt{\frac{\mu}{\lambda}}$, as $x \to +\infty$, so that the soliton interpolates smoothy between these vacua. The other vacuum is negative, namely, $\phi_{\mathrm{vacuum1}}(x) = -\sqrt{\frac{\mu}{\lambda}}$.

The classical stability of the soliton is ensured by considering small perturbations around it:

$$\phi(x, t) = \phi_s(x) + \eta(x, t). \tag{6}$$

Next, making the standard expansion of the fluctuations in terms of the normal modes,

$$\eta(x, t) = \sum_n \epsilon_n \eta_n(x) \mathrm{e}^{\mathrm{i}\omega_n t} \tag{7}$$

where ϵ_n are chosen so that $\eta_n(x)$ are real, the equation of motion becomes a Schrödinger-like equation in the supersymmetric form:

$$\left\{-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W^2(x) + W'(x)\right\}\eta_n(x) = {\omega_n}^2\eta_n(x) \tag{8}$$

where the superpotential is given by

$$W(x) = \lambda\left(3\phi_s^2 - \frac{\mu}{\lambda}\right) = \frac{\mu}{2}(3\tanh\mu x + 1) \tag{9}$$

and the prime means a first derivative with respect to the argument. The corresponding potential is given by

$$V_-(x) = W^2 + \frac{\mathrm{d}}{\mathrm{d}x}W(x) = -\frac{5}{4}\mu^2 + \frac{3}{2}\mu^2\tanh\mu x + \frac{15\mu^2}{4}\tanh^2\mu x. \tag{10}$$

The SUSY Hamiltonian in its bilinear form can be readily realized as

$$H_{\mathrm{SUSY}} = \begin{pmatrix} H_{ss}^- & 0 \\ 0 & H_{ss}^+ \end{pmatrix} = \begin{pmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{pmatrix}. \tag{11}$$

The bosonic sector Hamiltonian (H_{ss}^-) satisfies the following anihilation condition for the ground state $A^-\eta_-^{(0)} = 0$, $\eta_-^{(0)} = \eta_-^{(0)}(x)$:

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mu}{2}(3\tanh\mu x + 1)\right)\eta_-^{(0)} = 0 \Rightarrow \eta_-^{(0)}(x) = N_-(1 - \tanh\mu x)\sqrt{1 + \tanh\mu x} \tag{12}$$

where $N_- = \sqrt{\frac{\mu}{2}}$ is the normalization constant. This ground state is exactly the eigenfunction of the translational mode of (8), namely, $\eta_-^{(0)}(x) \equiv \eta_0(x) = \frac{\mathrm{d}}{\mathrm{d}x}\phi_s(x)$.

On the other hand the fermionic sector Hamiltonian(H_{ss}^+) does not have the zero mode because $\eta_+^{(0)}$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mu}{2}(3\tanh\mu x + 1)\right)\eta_+^{(0)} = 0 \Rightarrow \eta_+^{(0)}(x) = N_+\frac{\sqrt{1 + \tanh\mu x}}{\mathrm{sech}^2\,\mu x} \tag{13}$$

is not normalizable. The eigenvalue equations for the supersymmetric partners $H_{ss}^{\mp}$ are equivalent to those studied in detail in [18].

3. New potential in QM

The principal motivation of this section is to point out the possibility of constructing a new isospectral potential in QM so as to lead us to deduce new potentials in classical field theories. Let us now consider a general unnormalizable eigenfunction $\eta_G(x)$ of H_{ss}^+ given by

$$\begin{aligned}\eta_G(x) &= \eta_+^{(0)}(x)\left\{\alpha + \int_{-\infty}^{x}[\eta_+^{(0)}(\tilde{x})]^{-2}\,\mathrm{d}\tilde{x}\right\} \\ &= N_+\frac{\sqrt{1 + \tanh\mu x}}{\mathrm{sech}^2\,\mu x}\left\{\alpha + \frac{1}{2}\left(\tanh\mu x - \frac{1}{2}\tanh^2\mu x\right)\right\}\end{aligned} \tag{14}$$

where α is an arbitrary parameter and $\eta_+^{(0)}$ is given by equation (13). A refactorization of the fermionic sector Hamiltonian H_{ss}^+ considered now as a new supersymmetric partner

$$\tilde{H}_{ss}^+ = B^-B^+ = A^-A^+ = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \tilde{V}_+(x) \tag{15}$$

with

$$\tilde{V}_+(x) = V_+(x) \qquad B^+\eta_G(x) = 0 \tag{16}$$

is effected.

By means of a successive SUSY transformation one can construct the supersymmetric partner $\tilde{H}^-_{ss} = B^+B^-$ of $\tilde{H}^+_{ss}$, which leads to a class of new isospectral potentials characterized by an arbitrary parameter α and $\mu > 0$:

$$\begin{aligned}
\tilde{V}_-(x) &= W^2 + W' + \frac{f(x)}{\gamma(x)} - \frac{g^2(x)}{\gamma^2(x)} \\
\gamma(x) &= 2\alpha + \tanh\mu x - \tfrac{1}{2}\tanh^2\mu x \\
g(x) &= \mu(\tanh^2\mu x - 1)(\tanh\mu x - 1) \\
f(x) &= 2\mu^2(\operatorname{sech}^2\mu x\tanh\mu x(\tanh\mu x - 1) - \operatorname{sech}^4\mu x).
\end{aligned} \tag{17}$$

Note that if $\alpha \neq \frac{1}{4}$ and $\alpha \neq \frac{3}{4}$, $\tilde{V}_-(x)$ is non-singular respectively as $x \to +\infty$ and $x \to -\infty$. All eigenvalues of $\tilde{H}^-_{ss}$ are eigenvalues of H^-_{ss}, but all eigenvalues other than that of the ground state of $\tilde{H}^-_{ss}$ are also eigenvalues of $\tilde{H}^+_{ss}$. From (14), (15) and (17) and the annihilation condition

$$B^-\tilde{\eta}^{(0)}_-(\tilde{x}) = 0 \qquad B^- = (B^+)^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\ell n\left\{\gamma(x)\frac{\sqrt{1+\tanh\mu x}}{\operatorname{sech}^2\mu x}\right\} \tag{18}$$

we find that

$$\tilde{\eta}^{(0)}_-(x) = \frac{\tilde{N}_-}{\gamma(x)}\frac{\operatorname{sech}^2\mu x}{\sqrt{1+\tanh\mu x}} \tag{19}$$

which is a normalizable eigenstate of $\tilde{H}^-_{ss}$ associated with the zero bosonic mode ($\omega^{(0)}_- = \omega_0 = 0$), where $\gamma(x)$ is given in (17) and the normalization constant becomes

$$\tilde{N}_- = \frac{2}{\sqrt{5}}\sqrt{\left(2\alpha - \frac{3}{2}\right)\left(\alpha + \frac{1}{2}\right)}. \tag{20}$$

Note that this normalization constant is independent of μ and $\tilde{\eta}^{(0)}_-(x)$ is only non-vanishing for $\alpha \leqslant -\frac{1}{2}$ or $\alpha \geqslant \frac{3}{4}$, for $\tilde{N}_- \geqslant 0$. When $\alpha = 0$ and $x = 0$ the solution $\tilde{\eta}^{(0)}_-(x)$ is singular and therefore cannot be the ground eigenstate of $\tilde{H}^-_{ss}$.

In figures 1 and 2 we plot the potentials $V_-(x)$ and $\tilde{V}_-(x)$ respectively. Note that in the region with $\alpha \leqslant -\frac{1}{2}$ and $\mu = 2$ we obtain a well potential around the interval $-0.9 < x < 0.1$. We see that both the potentials $V_-(x)$ and $\tilde{V}_-(x)$ tend to finite values asymptotically equal as the Rosen–Morse potential. In figure 3 we plot the curves representing $\tilde{\eta}^{(0)}_-(\tilde{x})$ for $\mu = 2, \alpha = 10$ and $\alpha = -10$ and $-\tilde{\eta}^{(0)}_-(\tilde{x})$ for $\mu = 2, \alpha = -10$. The two curves are quite similar since for $|\alpha| \gg 1$ an approximate form for $\tilde{\eta}^{(0)}_-(\tilde{x})$ is given by

$$\tilde{\eta}^{(0)}_-(\tilde{x}) = \left(\frac{2}{5}\right)^{\frac{1}{2}}\operatorname{sgn}\alpha\frac{\operatorname{sech}^2\mu x}{\sqrt{1+\tanh\mu x}}. \tag{21}$$

The corresponding density of probability for $\mu = 2$ and $\alpha = \pm 10$ is shown in figure 4.

4. Double-well like potentials in 1 + 1 dimensions

From the new potential of $\tilde{H}^-_{ss}$ can be obtained a new class of potential nonlinear models $\tilde{V}(\phi)$ in 1 + 1 dimensions. Equations (5) and (17) together define the potential $\tilde{V}(\phi)$ as a function of

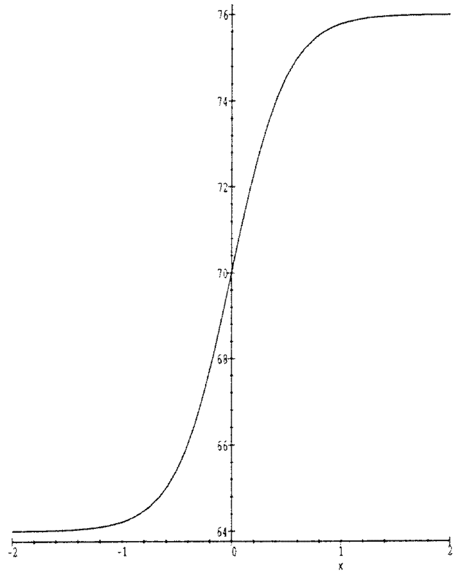

Figure 1. The potential $V_{-}(x)$, for $\mu = 2$.

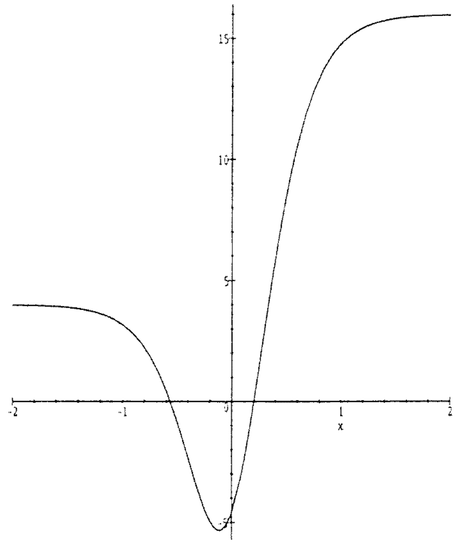

Figure 2. The potential $\tilde{V}_{-}(x)$, for $\mu = 2$ and $\alpha = -10$.

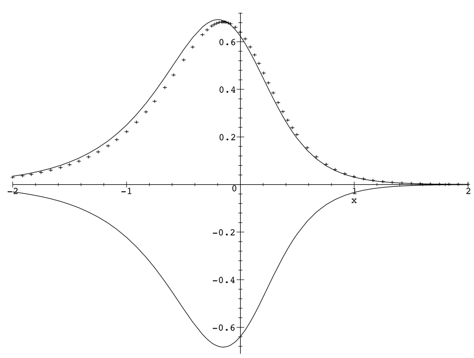

Figure 3. The wavefunction for the ground state of $\frac{\tilde{\eta}_{-}^{(0)}(\tilde{x})}{817}$ of $\tilde{V}_{-}(x)$, for $\mu = 2$ and $\alpha = \pm 10$ and $-\tilde{\eta}_{-}^{(0)}(\tilde{x})$, for $\mu = 2$, $\alpha = -10$, corresponding to the dotted curve.

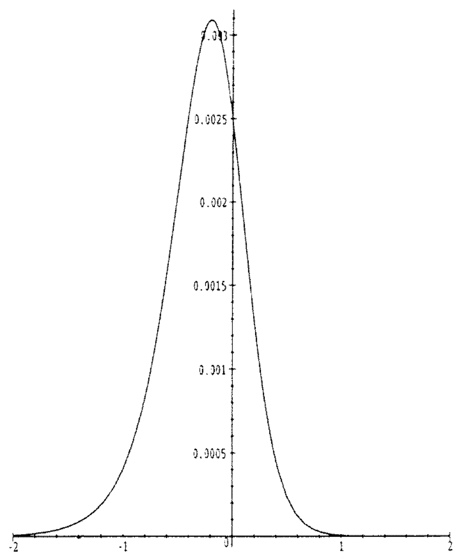

Figure 4. The density of probability $5.76 \times 10^{-3}(\tilde{\eta}_{-}^{(0)}(\tilde{x})^2$, for $\mu = 2$ and $\alpha = \pm 10$.

ϕ. Indeed, from equation (5) one gets $x = x(\phi)$ so that the explicit form of the new potential in field theory in 1 + 1 dimensions is given below.

Let us now consider such a new class of potentials in 1 + 1 dimensions as a function of

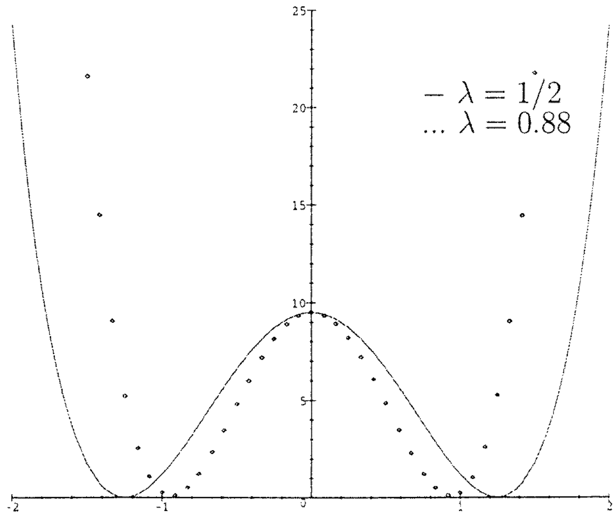

Figure 5. The potential ($\tilde{V}(\phi) + 5.5$), for the particular case of $\lambda = \frac{1}{2}, 0.88$.

the real field ϕ as given by:

$$\tilde{V}(\phi; m, \lambda, \alpha) = \frac{m^2\theta^2 + a\phi^2 + b\phi^4 + c\phi^6 + d\phi^8 + e\phi^{10} + p\phi^{12}}{(\theta + 8\lambda m\phi^2 - 4\lambda^4\phi^4)^2} \tag{22}$$

where

$$\begin{aligned} a &= m^5\lambda(480\alpha - 192\alpha^2 - 252) \\ b &= m^4\lambda^2(1375 + 240\alpha^2 - 1928\alpha) \\ c &= \lambda^3 m^3(2496\alpha - 3344) \\ d &= m^2\lambda^4(4024 - 992\alpha) \\ e &= -2304m\lambda^5 \qquad p = 496\lambda^6 \\ \theta &= m^2(4\alpha - 3). \end{aligned} \tag{23}$$

Equations (22) and (23) are obtained as follows. Using equation (5) for $x_0 = 0$ one may put

$$x = \frac{1}{\mu}\tanh^{-1}\left(\frac{2\lambda\phi^2}{\mu} - 1\right). \tag{24}$$

When we substitute this value for the position coordinate in the new potential $\tilde{V}_-(x)$ given by equation (17) it is exactly changed to $\tilde{V}(\phi; m, \lambda, \alpha)$, after some algebra as shown above.

The potential $\tilde{V}(\phi; m, \lambda, \alpha)$ possesses a discrete symmetry $\phi \to -\phi$, which must have at least two different zeros in order to present solitons as solutions. From (22) and condition $\lambda > 0$ we see that the potential $\tilde{V}(\phi)$ is nonsingular if and only if $0 < \lambda^2 < \frac{1}{\frac{3}{4} - \alpha}$. When $\alpha < -\frac{1}{2}$ this condition becomes $0 < \lambda < 0.89$ and $\theta < -5m^2$ so that in this case the potential $\tilde{V}(\phi)$ possesses degenerate minima values and a maximum value. This shows that the curve has an analogous behaviour as the double-well potential of the kink of the $\lambda\phi^4$ theory. This may be seen from figure 5 when we shift the potential ($\tilde{V}(\phi) + 5.5$), for the particular case in that $\alpha = -10$, $m = 2$, $-2 < \phi < 2$ and $\lambda = \frac{1}{2}, \frac{1}{100}$ respectively.

Note that in the region $\alpha < -\frac{1}{2}$ and around the upper limit for $\lambda = \frac{88}{100}$ the double-well like potential has other values for the vacua and the curve is in the interval $-2 < \phi < 2$.

5. Conclusion

The connection between the soliton solutions of the ϕ^6 model in 1 + 1 dimensions and SUSY QM has been shown. From the stability equation of such a soliton, which is a Schrödinger-like equation for the normal modes associated with the potential $V_-(x)$, a new class of potentials $\tilde{V}_-(x)$ in QM and $\tilde{V}(\phi; m, \lambda, \alpha)$ as a function of real field ϕ in the bidimensional spacetime was constructed. A generalization was found from the potential of ϕ^6 model. We found that both the potentials $V_-(x)$ and $\tilde{V}_-(x)$ are Rosen–Morse like potentials. Other techniques can be implemented in order to construct a new class of potentials in 1 + 1 dimensions [14].

According to the above analysis for the double-well potential we see that from the stability equation for the potential $\tilde{V}(\phi; m, \lambda, \alpha))$ we can find the following Schrödinger-like potential:

$$\tilde{V}_- = \tilde{V}_-(x; m, \alpha)$$

whose explicit dependence is guaranteed if it were possible to solve the equation of motion for the new class of potentials. In this case the SUSY QM may be considered and, therefore, the new potential $\tilde{V}(x; m, \alpha)$ of $H_{ss}^- = -\frac{d^2}{dx^2} + \tilde{V}_-$ can be found as another class of potential nonlinear models in field theory in the bidimensional spacetime, in a way analogous to that recently implemented for the soliton of the $\lambda\phi^4$ theory [4]. Other approaches have also been independently treated in the literature [14].

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